

Maximal Coherence for Prescribed Intrinsic Populations and Youla Values: Two Questions of Gil

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Abstract

In the intrinsic reference basis of a finite-dimensional density matrix, $\rho = A + iN$ with $A = \text{diag}(a_1 \geq \dots \geq a_n \geq 0)$ the intrinsic populations and N real antisymmetric; Gil (Entropy 28:877, 2026) calls $\|N\|_{\mathbb{F}}^2$ the squared cohesion. Gil showed that among states whose metaspin tensor is supported on a matching of intrinsic axes (the aligned class) the squared cohesion is maximized, for fixed populations, by pairing adjacent populations, and asked whether this remains true over all orientations of the Youla frame with free Youla values, and what the maximum over all orientations is when the Youla values are prescribed. We determine both maxima. With free Youla values the maximum over all states with these populations is $2 \sum_k a_{2k-1} a_{2k}$, so Gil's adjacent-pairing value is the global maximum. With prescribed Youla values $1 \geq s_1 \geq \dots \geq s_m \geq 0$, $m = \lfloor n/2 \rfloor$, the maximum over all orientations is $2 \sum_k s_k^2 a_{2k-1} a_{2k}$, attained by placing the k -th largest Youla value on the k -th adjacent pair; when populations vanish, both statements are read on their support. The second value rests on the inequality $\sum_{i < j} a_i a_j |M_{ij}|^2 \leq \sum_k s_k^2 a_{2k-1} a_{2k}$ for every skew-symmetric M with singular values $s_1, s_1, \dots, s_m, s_m$, which follows from Horn's 1950 inequality for the singular values of a product and is the Frobenius case of Mathias's 1992 inequality for Hadamard products; we claim no priority for it. Two proofs are given. For free values, the squared entries of a real antisymmetric contraction lie in Edmonds' matching polytope, which extends the statement to arbitrary nonnegative pairwise weights; for prescribed values, we use a layer-cake decomposition with Ky Fan's maximum principle and interlacing of singular values, which yields a family of linear inequalities on the squared entries of skew matrices with prescribed spectrum. The product form of the weights is essential there: with prescribed Youla values and general nonnegative weights the maximum need not be aligned, as an explicit example with $n = 4$ shows. Gil's question on the image of the population-weighted Gram map, and the feasibility problem with the spectrum of ρ fixed, are not addressed.

Attribution

This paper was written by Claude Fable 5.1, an AI model made by Anthropic, at the direction of David Ross. The model found the two questions of Gil while reviewing the companion manuscript on approximate antiunitary symmetry (whose hull lemma it reuses), proved the theorems, wrote the verification program and the text; David Ross set the task, ran the process, and takes responsibility for the manuscript. The answer to the free-value question (Theorem 3.1, eq. (7)) was also found and proved independently, through the same hull lemma, by the companion manuscript's adversarial

self-review (GPT-6 Astra, 2026-09-28), before this note's first draft; neither review had read the other, and the companion's Section 1 records that both found the application. Four reviews were applied, each with a per-item ledger. On 2026-09-29, Claude Opus 5.5 (Anthropic), in a fresh context, found the proofs complete and three framing defects (an overclaimed abstract, refuted by the $n = 4$ example it supplied, now Remark 4.5; an unread source behind a novelty claim; a misreported numerical search); GPT-6 Astra (OpenAI), through the Codex CLI, found no blocking defect and two major items (Theorem 4.1 follows from Horn's inequality, now its second proof; Corollary 5.1(1) needed its support restriction) and supplied the exact certificates of Remark 4.5. A consistency pass on 2026-10-01 by Claude Opus 5.5 checked the exposition and the quotations against their sources. On October 2, GPT-6 Astra, fresh to the note, found both theorems and the hull lemma correct and corrected the exact-orbit certificate of Remark 4.5 (a factor of two and the sign of the Pfaffian; the stated maximum unchanged) and a quotation locator; Claude Opus 5.5, fresh to the note, found every numbered statement correct and two major items of attribution and source access (the earlier independent proof of Theorem 3.1; the disclosure about Mathias's abstract) and ten minor ones. All were applied. A literature pass the same day found Mathias's 1992 theorem, which contains inequality (9) as its Frobenius case; Remark 4.4 cites it through its zbMATH review. The reviews and the per-item dispositions accompany the source. David Ross read every page, for the only things he can judge: that the account of the harness and of the process matches the record, and that nothing reads like a machine grading its own homework. He did not check the proofs, and could not have; a reader should take his reading as a check for red flags, not as a review. No human mathematician has reviewed this paper. It is one of three manuscripts that came from the record of Hypnos, the research harness described in Section 7, on the night of 2026-09-28/29, this one at one remove (the harness contributed no step of its mathematics); it is the last of the three, after Astra's antiunitary paper and the Riesz-basis paper on the unfolded zeta zeros, and it was written without reading the latter.

1 The question

Gil [1] organizes a finite-dimensional density matrix ρ by diagonalizing its real part with a real orthogonal congruence. In the resulting intrinsic reference basis (IRB),

$$\rho = A + iN, \quad A = \text{diag}(a_1, \dots, a_n), \quad a_1 \geq \dots \geq a_n \geq 0, \quad \sum_i a_i = 1, \quad N^T = -N, \quad (1)$$

with N real (Gil, eq. (3)); a density matrix is Hermitian, positive semidefinite and trace-normalized (Gil, Section 2.1), and the IRB is obtained from a fixed real structure by a real orthogonal change of basis, so the a_i , the intrinsic populations, are the eigenvalues of the real part of ρ and not in general those of ρ . For strictly positive populations the metaspin tensor is

$$M_{ij} = \frac{N_{ij}}{\sqrt{a_i a_j}} \quad (i \neq j) \quad (2)$$

(Gil, eq. (7)), a real antisymmetric matrix; when some populations vanish, M is defined on the support $\{i : a_i > 0\}$ and extended by zeros to the inactive axes (Gil, Section 2.3), so its entries on inactive axes are not free variables. Then

$$\rho = A^{1/2}(I + iM)A^{1/2} \quad (3)$$

(Gil, eq. (8)). Since iM is Hermitian with eigenvalues $\pm s_k$ (and 0 when n is odd), where $s_1 \geq s_2 \geq \dots \geq s_m \geq 0$ ($m = \lfloor n/2 \rfloor$) are the Youla values of M , positivity of ρ is equivalent to $s_1 \leq 1$, that

is, to

$$\|M\|_{\text{op}} \leq 1. \quad (4)$$

Gil calls $\|N\|_{\text{F}}$ the cohesion of the antisymmetric block; its square is

$$\|N\|_{\text{F}}^2 = 2 \sum_{i < j} a_i a_j M_{ij}^2 \quad (5)$$

(Gil, eqs. (20) and (33)), and it enters his complementarity identities for purity, spectral sharpening and Rényi entropies. By the real normal form of a real skew-symmetric matrix under orthogonal congruence (Youla [3] for the complex case under unitary congruence; [8, Section 2.5] for the real case) there is $Q_M \in \text{O}(n)$ with $Q_M^T M Q_M = \bigoplus_k \begin{pmatrix} 0 & s_k \\ -s_k & 0 \end{pmatrix} \oplus 0$; Gil calls ρ aligned (his Definition 1) when, in some admissible IRB representative, Q_M can be taken to be a signed permutation of the IRB axes, so that M is supported on a matching of index pairs.

Theorem 1.1 (Gil [1], Theorem 2). *For fixed ordered populations, the maximum of $\|N\|_{\text{F}}^2$ over the aligned class is $2 \sum_{k=1}^n a_{2k-1} a_{2k}$, attained by the adjacent pairing $(1, 2), (3, 4), \dots$ with $s_k = 1$ on every pair of positive population product.*

Gil then writes, in the paragraph after his Theorem 2 (p. 10 of the publisher’s PDF, where the section is numbered 3.5; the PubMed Central HTML rendering numbers it 3.4): “Whether the same maximum is globally optimal over arbitrary Youla orientations is a separate optimization problem and is not assumed here,” and lists in Section 8.2 three problems “deliberately left open because they require optimization over Grassmannian orbits rather than the elementary matching argument used here”; they are unnumbered there, and we number them (i) to (iii): (i) to determine the global maximum of $\|N\|_{\text{F}}^2$ for fixed populations and fixed Youla values over all orientations Q_M ; (ii) to characterize the image of the population-weighted Gram map of Plücker coordinates; (iii) to decide whether the adjacent-pairing maximum of Theorem 1.1 remains globally optimal beyond the aligned class under suitable spectral constraints.

We determine the maximum asked for in (i) (Theorem 4.1) and answer (iii) affirmatively (Theorem 3.1); (ii) is not addressed. In (iii) we read “suitable spectral constraints” as constraints on the Youla values: if the spectrum of ρ were fixed together with the populations, $\text{tr } \rho^2 = \sum_i a_i^2 + \|N\|_{\text{F}}^2$ would already fix the objective, and the remaining question, whether an aligned representative is compatible with such a restriction on the spectrum of ρ , would be a feasibility question that we do not address. For free Youla values the Grassmannian orbit never has to be parametrized: the squared entries of every orientation lie in Edmonds’ matching polytope, so the maximization becomes linear optimization over a polytope whose vertices are matchings. For prescribed values the product-weight maximum follows instead from the singular-value inequality of Theorem 4.1.

2 Squared entries of an antisymmetric contraction

Let $\mathcal{M}_n$ be the set of matchings of $\{1, \dots, n\}$ (sets of disjoint pairs, the empty matching included), $\mathbf{1}_\mu \in \mathbb{R}^{\binom{n}{2}}$ the incidence vector of a matching μ (the letter M is reserved for the metaspin tensor), $E(B)$ the pairs inside a vertex set B , and $\delta(i)$ the pairs containing i . Edmonds [2] proved that the convex hull of the incidence vectors is

$$\mathcal{P}_n = \left\{ x \geq 0 : \sum_{e \in \delta(i)} x_e \leq 1 \ (1 \leq i \leq n), \quad \sum_{e \in E(B)} x_e \leq \frac{|B|-1}{2} \ (|B| \text{ odd}) \right\}. \quad (6)$$

Lemma 2.1 (squared-entry hull, from [13]). *Let K be an $n \times n$ real or complex skew-symmetric matrix with $\|K\|_{\text{op}} \leq 1$ and put $x_{\{i,j\}} = |K_{ij}|^2$ for $i < j$. Then $x \in \mathcal{P}_n$. Conversely every $\mathbf{1}_\mu$ is of this form, so the convex hull of the set of such vectors is exactly $\mathcal{P}_n$.*

Proof. Nonnegativity is clear. The i -th row of a contraction has squared norm at most 1, which gives the degree inequalities. For $|B|$ odd, the principal submatrix K_B is skew-symmetric of odd order, so $\det K_B = \det K_B^T = \det(-K_B) = -\det K_B$, hence $\det K_B = 0$ and $\text{rank } K_B \leq |B| - 1$. K_B is a contraction, so each of its singular values is at most 1 and $2 \sum_{e \in E(B)} x_e = \|K_B\|_F^2 \leq \text{rank } K_B \leq |B| - 1$. Thus x satisfies (6). Conversely, the matrix with block $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on each pair of μ and zeros elsewhere is a skew-symmetric partial isometry, hence a contraction, with squared entries $\mathbf{1}_\mu$. Since $\mathcal{P}_n$ is convex and contains all the vectors, and its vertices are among them, the hull identity follows. $\square$

This lemma is taken from the companion manuscript [13], where it drives an exact reduction of approximate antiunitary symmetry to weighted matching; its application to density matrices was found independently by both reviews of that manuscript (its Section 1 says so) and is developed here. The set of squared-entry vectors itself is not convex (a strict mixture of two matchings need not be realized), and no point violating an odd-set inequality is realized by any contraction; the hull statement is what matters below.

3 Free Youla values

Theorem 3.1. *Let $a_1 \geq \dots \geq a_n \geq 0$ with $\sum_i a_i = 1$ be fixed. Over all density matrices whose intrinsic representation (1) has these populations, with every orientation of the Youla frame and every admissible Youla spectrum,*

$$\max \|N\|_F^2 = 2 \sum_{k=1}^m a_{2k-1} a_{2k}, \quad (7)$$

attained inside the aligned class by the adjacent pairing with $s_k = 1$ on every adjacent pair of positive population product. More generally, if all populations are positive, then for any nonnegative weights b_{ij} ,

$$\max \sum_{i < j} b_{ij} M_{ij}^2 = \max_{\mu \in \mathcal{M}_n} \sum_{\{i,j\} \in \mu} b_{ij}, \quad (8)$$

the maximum weight of a matching, and the set of admissible vectors $(M_{ij}^2)_{i < j}$ has convex hull $\mathcal{P}_n$; if some populations vanish, both statements hold with the matchings and the polytope of the support of A .

Proof. Suppose first that all populations are positive. By (3) and (4), the admissible coherence blocks are exactly $N = A^{1/2} M A^{1/2}$ with M a real antisymmetric contraction, and by (5) the squared cohesion is $2 \sum_{i < j} a_i a_j x_{ij}$ with $x = (M_{ij}^2)$. By Lemma 2.1, $x \in \mathcal{P}_n$. A linear functional with nonnegative coefficients attains its maximum over the polytope at a vertex $\mathbf{1}_\mu$, with value $2 \sum_{\{i,j\} \in \mu} a_i a_j$. Among matchings the adjacent pairing is optimal. First complete μ to a matching of maximal size, which does not decrease the weight since the weights are nonnegative. If n is odd and the unpaired vertex is $v < n$, give v the partner of n and leave n unpaired; the weight does not decrease since $a_v \geq a_n$. Now, for $a_i \geq a_j \geq a_k \geq a_l$,

$$\begin{aligned} a_i a_j + a_k a_l - (a_i a_k + a_j a_l) &= (a_i - a_l)(a_j - a_k) \geq 0, \\ a_i a_j + a_k a_l - (a_i a_l + a_j a_k) &= (a_i - a_k)(a_j - a_l) \geq 0, \end{aligned}$$

so any crossing or nested pair of pairs can be uncrossed without decreasing the weight; these are the exchange inequalities (26), (27) of [1]. Each uncrossing of pairs $\{i, k\}, \{j, l\}$ or $\{i, l\}, \{j, k\}$ with $i < j < k < l$ into $\{i, j\}, \{k, l\}$ lowers $\sum_{\{i, j\} \in \mu} |i - j|$ by $2(k - j) > 0$, so the process stops, and the only perfect matching of $\{1, \dots, 2m\}$ without a crossing or nested pair is the adjacent pairing. The adjacent signed swap with $s_k = 1$ is an antisymmetric contraction, so the value is attained, and the corresponding state is aligned; explicitly, it is block diagonal with the blocks

$$\begin{pmatrix} a_{2k-1} & i\sqrt{a_{2k-1}a_{2k}} \\ -i\sqrt{a_{2k-1}a_{2k}} & a_{2k} \end{pmatrix},$$

of eigenvalues $a_{2k-1} + a_{2k}$ and 0, followed by the unpaired population when n is odd: Hermitian, positive semidefinite, of trace one, with real part A and $\|N\|_F^2 = 2\sum_k a_{2k-1}a_{2k}$. The same argument gives (8) for arbitrary nonnegative weights, and the hull statement is the second half of Lemma 2.1. If some populations vanish, positivity forces the corresponding rows and columns of N to vanish ([1], Section 2.3), so M is defined on the support of A and the argument applies there; the zero populations contribute nothing to (7), in agreement with the formula, while (8) and the hull statement must be read on the support: for $a = (\frac{1}{2}, \frac{1}{2}, 0, 0)$, $b_{12} = 1$, $b_{34} = 10$, the maximum over states is 1 and the unrestricted matching weight is 11. $\square$

Theorem 3.1 answers question (iii): under the only constraint Gil imposes, positivity, the adjacent-pairing maximum of Theorem 1.1 is the global maximum, and the aligned class contains a maximizer of every nonnegative pairwise coherence functional. The value (7) also follows from Mathias's inequality with $\|M\|_{\text{op}} \leq 1$ (Remark 4.4); what the matching polytope adds is the general-weight statement (8) and the hull.

4 Prescribed Youla values

The inequality below follows from Horn's product inequality [5] and is the Frobenius case of a theorem of Mathias [10], known to us through its zbMATH review; Remark 4.4 gives both deductions.

Theorem 4.1. *Let $a_1 \geq \dots \geq a_n \geq 0$, $m = \lfloor n/2 \rfloor$, and $s_1 \geq \dots \geq s_m \geq 0$. Let M be any real or complex skew-symmetric $n \times n$ matrix whose singular values are $s_1, s_1, \dots, s_m, s_m$ (and 0 if n is odd); equivalently $M = Q\Sigma_s Q^T$ with Q orthogonal or unitary and Σ_s the Youla block form. Then*

$$\sum_{i < j} a_i a_j |M_{ij}|^2 \leq \sum_{k=1}^m s_k^2 a_{2k-1} a_{2k}, \quad (9)$$

with equality for $Q = I$. Consequently, for fixed populations and fixed Youla values with $s_1 \leq 1$, the global maximum of $\|N\|_F^2$ over all orientations Q_M is $2\sum_k s_k^2 a_{2k-1} a_{2k}$, attained in the aligned class by placing the k -th largest Youla value on the k -th adjacent population pair. (For positive populations every orientation with $s_1 \leq 1$ is realized by a state. If $r < n$ populations are positive, a state exists only when at most $\lfloor r/2 \rfloor$ of the s_k are positive, and the statement is read on the support of A .)

Proof. Write $x_{ij} = |M_{ij}|^2$. For nonnegative reals, $a_i a_j = \int_0^\infty \int_0^\infty \mathbf{1}[a_i > u] \mathbf{1}[a_j > v] du dv$, and averaging this with the same identity in the opposite order,

$$\sum_{i < j} a_i a_j x_{ij} = \iint \Psi(u, v) du dv, \quad \Psi(u, v) = \frac{1}{2} \sum_{i < j} x_{ij} (\mathbf{1}[a_i > u, a_j > v] + \mathbf{1}[a_i > v, a_j > u]). \quad (10)$$

Likewise the right side of (9) equals $\iint h(u, v) du dv$ with $h(u, v) = \frac{1}{2} \sum_k s_k^2 (\mathbf{1}[a_{2k-1} > u, a_{2k} > v] + \mathbf{1}[a_{2k-1} > v, a_{2k} > u])$. Both integrands are symmetric in (u, v) , so it suffices to prove $\Psi(u, v) \leq h(u, v)$ when $u \geq v$. Fix such (u, v) and let $C = \{i : a_i > u\}$ and $B = \{i : a_i > v\}$. Because a is sorted these are prefix sets, $C = \{1, \dots, p\} \subseteq B = \{1, \dots, q\}$.

The integrand Ψ . For $i < j$ we have $a_i \geq a_j$, hence $\mathbf{1}[a_i > u, a_j > v] = \mathbf{1}[i \in C, j \in B]$ and $\mathbf{1}[a_i > v, a_j > u] = \mathbf{1}[j \in C]$ (if $j \in C$ then $i \in C \subseteq B$). The pairs of B whose smaller index lies in C are the pairs inside C together with the pairs between C and $B \setminus C$, so

$$2\Psi(u, v) = 2x(E(C)) + x(C, B \setminus C),$$

where $x(C, B \setminus C)$ denotes the mass on pairs with one end in C and the other in $B \setminus C$. In matrix terms $2x(E(C)) = \|M_C\|_F^2$ and $x(C, B \setminus C) = \|M_{C, B \setminus C}\|_F^2$, whence

$$\Psi(u, v) = \frac{1}{2} \|M_{C, B}\|_F^2, \quad (11)$$

one-half the squared Frobenius norm of the $p \times q$ block of M with rows C and columns B .

The integrand h . $\mathbf{1}[a_{2k-1} > u, a_{2k} > v] = \mathbf{1}[2k-1 \leq p, 2k \leq q]$ and $\mathbf{1}[a_{2k-1} > v, a_{2k} > u] = \mathbf{1}[2k \leq p]$. Counting,

$$h(u, v) = \frac{1}{2} R(p, q), \quad R(p, q) = \sum_{k \leq \min(\lceil p/2 \rceil, \lfloor q/2 \rfloor)} s_k^2 + \sum_{k \leq \lfloor p/2 \rfloor} s_k^2. \quad (12)$$

The inequality $\|M_{C, B}\|_F^2 \leq R(p, q)$. The block $M_{C, B}$ consists of p rows of the principal submatrix M_B , which is skew-symmetric of order q ; its singular values come in pairs $\tau_1 \geq \tau_1 \geq \tau_2 \geq \tau_2 \geq \dots \geq \tau_r \geq \tau_r$, $r = \lfloor q/2 \rfloor$, with an additional 0 if q is odd. Two classical facts apply. First, for any matrix X and the coordinate projection P onto p of its rows, $\|PX\|_F^2 = \text{tr}(PXX^*P) \leq \sum_{k \leq p} \sigma_k(X)^2$ (Ky Fan's maximum principle [4, 6]). Second, the singular values of a submatrix are dominated termwise by those of the matrix, $\sigma_k(M_B) \leq \sigma_k(M)$ ([7, Thm. 3.1.2, Cor. 3.1.3]), which, both sequences being sorted decreasingly, gives $\tau_k \leq s_k$. Hence $\|M_{C, B}\|_F^2 \leq \sum_{k \leq p} \sigma_k(M_B)^2$, and we compare this sum with $R(p, q)$ in the three possible cases.

- p even: $p/2 \leq \lfloor q/2 \rfloor$, so $\sum_{k \leq p} \sigma_k(M_B)^2 = 2 \sum_{k \leq p/2} \tau_k^2 \leq 2 \sum_{k \leq p/2} s_k^2 = R(p, q)$.
- p odd and $p < q$: then $(p+1)/2 \leq \lfloor q/2 \rfloor$, so $\sum_{k \leq p} \sigma_k(M_B)^2 = 2 \sum_{k \leq (p-1)/2} \tau_k^2 + \tau_{(p+1)/2}^2 \leq 2 \sum_{k \leq (p-1)/2} s_k^2 + s_{(p+1)/2}^2 = R(p, q)$.
- $p = q$ odd: $\sum_{k \leq p} \sigma_k(M_B)^2 = 2 \sum_{k \leq r} \tau_k^2 \leq 2 \sum_{k \leq (q-1)/2} s_k^2 = R(p, q)$.

Thus $\Psi \leq h$ pointwise, and integrating (10) proves (9). For $Q = I$ the only nonzero entries are $M_{2k-1, 2k} = -M_{2k, 2k-1} = s_k$, and the left side equals the right side. The consequence for the squared cohesion follows from (5) and the fact that the aligned configuration is admissible; for the density-matrix equality case replace the off-diagonal entries of the blocks in the proof of Theorem 3.1 by $\pm i s_k \sqrt{a_{2k-1} a_{2k}}$, whose determinant $a_{2k-1} a_{2k} (1 - s_k^2)$ is nonnegative. $\square$

Second proof, from a classical singular-value inequality. Put $D = \text{diag}(\sqrt{a_1}, \dots, \sqrt{a_n})$ and $T = DMD$. Then T is skew-symmetric, so its singular values are $t_1, t_1, \dots, t_m, t_m$ (and 0 when n is odd), and $\sum_{i < j} a_i a_j |M_{ij}|^2 = \frac{1}{2} \|T\|_F^2 = \sum_k t_k^2$. Horn's inequality for the singular values of a product [5] states that $\prod_{j \leq r} \sigma_j(XY) \leq \prod_{j \leq r} \sigma_j(X) \sigma_j(Y)$ for every r . Applied twice at the even index $r = 2\ell$, with $\sigma_j(D) = \sqrt{a_j}$ in decreasing order,

$$\prod_{k \leq \ell} t_k^2 = \prod_{j \leq 2\ell} \sigma_j(DMD) \leq \left(\prod_{j \leq 2\ell} \sigma_j(D)^2 \right) \prod_{j \leq 2\ell} \sigma_j(M) = \prod_{k \leq \ell} s_k^2 a_{2k-1} a_{2k} \quad (1 \leq \ell \leq m).$$

Both $(t_k^2)_k$ and $(s_k^2 a_{2k-1} a_{2k})_k$ are nonincreasing, so this says that the first is weakly log-majorized by the second, which implies weak majorization (apply the increasing convex function $\exp$ to the logarithms; zeros are handled by continuity), and in particular $\sum_k t_k^2 \leq \sum_k s_k^2 a_{2k-1} a_{2k}$, which is (9). Equality for $Q = I$ is as before. $\square$

Theorem 4.1 answers question (i), and the second proof shows that, as a matrix inequality, it is a sharp consequence of classical majorization, and indeed a special case of Mathias's 1992 inequality (Remark 4.4); what the layer-cake proof adds is the family of Proposition 4.2. The theorem recovers (7): apply it to each contraction with its own Youla values and use that the right side of (9) is nondecreasing in every s_k . It does not recover (8) or the hull statement, which concern non-product weights; Remark 4.5 shows that the prescribed-value analog of (8) is false, so they cannot be obtained one Youla spectrum at a time.

Proposition 4.2 (valid inequalities for prescribed spectrum). *Let M be skew-symmetric with singular values $s_1, s_1, \dots, s_m, s_m$ (and 0 if n is odd), and $x_{ij} = |M_{ij}|^2$. For all vertex sets $C \subseteq B$,*

$$2 \sum_{e \in E(C)} x_e + \sum_{e \in E(B) \setminus E(C) \setminus E(B \setminus C)} x_e \leq R(|C|, |B|), \quad (13)$$

with R as in (12). For $|C| = 1$ and $|B| \geq 2$ this is the degree constraint $\sum_{e \in \delta(i) \cap E(B)} x_e \leq s_1^2$ (the singleton $C = B$ reads $0 \leq 0$); for $C = B$ it is the refined odd/even-set constraint $\sum_{e \in E(B)} x_e \leq \sum_{k \leq \lfloor |B|/2 \rfloor} s_k^2$, which for $s \equiv 1$ is Edmonds' odd-set inequality.

Proof. The proof of Theorem 4.1 never used that C and B are prefix sets: the left side of (13) is $\|M_{C,B}\|_F^2$ and the bound is the one just established. $\square$

Remark 4.3. The subfamily $C = B$ of (13) is not sufficient for Theorem 4.1: the linear program that maximizes $\sum_{i < j} a_i a_j x_{ij}$ subject only to those constraints exceeds the true value of that sum in random instances (by up to 0.0097 in the checks of Section 6, that is, 0.019 in units of $\|N\|_F^2$), and adding the degree constraints to $C = B$ does not close the gap (in 4 of the 30 instances the value is still exceeded, by up to $2.2 \cdot 10^{-4}$, or $4.4 \cdot 10^{-4}$ in units of $\|N\|_F^2$). Conversely, the proof uses only the $n(n+1)/2$ prefix inequalities $C = \{1, \dots, p\} \subseteq B = \{1, \dots, q\}$, and the linear program over those alone reproduces the value of Theorem 4.1 to machine precision in every instance, as does the linear program over the full family (13). Some mixed constraints with $C \subsetneq B$ are therefore essential, and the prefix ones suffice.

Remark 4.4. The zero diagonal of a skew matrix is what allows the product form. The generic bound for a Hermitian matrix X with eigenvalues μ_i , $\|A^{1/2} X A^{1/2}\|_F^2 = \text{tr}(A X A X) \leq \text{tr}(A^2 X^2) \leq \sum_i a_i^2 \mu_i^2$ (Cauchy–Schwarz, then von Neumann's trace inequality with both sequences sorted), applied to $X = iM$ gives only $\sum_k s_k^2 (a_{2k-1}^2 + a_{2k}^2)$, which is weaker than (9) by the arithmetic–geometric mean inequality and is attained by diagonal X , a form unavailable to iM . As a matrix inequality, (9) is not new in substance: the second proof of Theorem 4.1 derives it in a few lines from Horn's 1950 product inequality [5], and it is a special case of a theorem of Mathias found after this note was written and known to us through its zbMATH review (Zbl 0759.15011; the paper itself was not obtained), as follows. The skew-symmetric Schur–Horn theorem (Leite, Richa and Tomei [9]; [11, Prop. 3.10], which concerns the linear skew-diagonal projection of an orbitope) and the main theorem of Tam [12] describe the entries on one fixed matching under congruence, the moduli of the skew-diagonal entries being weakly majorized by the Youla values (with one further inequality, of Thompson–Sing type, in Tam's even-dimensional full-matching case); they do not directly state the weighted squared-entry inequality. Mathias [10] proved, as the review states it, that for a real

positive semidefinite A with ordered diagonal entries $\alpha_1 \geq \dots \geq \alpha_n$ and a complex skew-symmetric B , $\sum_{i \leq k} \sigma_i(A \circ B) \leq \sum_{i \leq k} \tilde{\alpha}_i \sigma_i(B)$ for every k , where $\tilde{\alpha}_{2i-1} = \tilde{\alpha}_{2i} = \sqrt{\alpha_{2i-1} \alpha_{2i}}$ (and $\tilde{\alpha}_n = 0$ for odd n), together with the corresponding bound in every unitarily invariant norm. With $A = dd^T$, $d_i = \sqrt{a_i}$, the Hadamard product is exactly the matrix $DMD = (dd^T) \circ M$ of the second proof, whose singular values are $t_1, t_1, t_2, t_2, \dots$, so his inequality says that $(t_k)_k$ is weakly majorized by $(\sqrt{a_{2k-1} a_{2k}} s_k)_k$, and the convexity of the square gives $\sum_k t_k^2 \leq \sum_k s_k^2 a_{2k-1} a_{2k}$, which is (9); with $\|M\|_{\text{op}} \leq 1$ it also gives the value (7). We claim no priority for (9): it is the Frobenius case of Mathias's theorem. Tam and Leite-Richa-Tomei were available to us only through their abstracts (Sanyal-Sottile-Sturmfels in full), and we do not claim that their papers contain no equivalent formulation. This note's contribution is the application to Gil's questions, through the hull lemma of the companion manuscript, and the nested-set family of Proposition 4.2; the searches described here do not establish priority for these formulations either.

Remark 4.5 (product weights are essential). With prescribed Youla values and general nonnegative weights b_{ij} the maximum of $\sum_{i < j} b_{ij} M_{ij}^2$ need not be attained in the aligned class, and the linear program over (13) need not be tight. Take $n = 4$ with distinct positive populations, say $a = (0.4, 0.3, 0.2, 0.1)$, so that alignment means matching support (distinct populations leave no residual rotation in Gil's Definition 1), $s = (1, 0.9)$, $b_{12} = 2$, $b_{13} = b_{24} = 1.2$, all other $b_{ij} = 0$, and $c = \sqrt{0.84}$. The real skew matrix with $M_{12} = 0.3$, $M_{13} = c$, $M_{24} = -c$, $M_{34} = 0.2$ (and zeros elsewhere above the diagonal) has $s_1^2 + s_2^2 = \frac{1}{2} \|M\|_F^2 = 1.81$ and $s_1 s_2 = |\text{Pf } M| = 0.9$, so its singular values are 1, 1, 0.9, 0.9; its value is $2(0.09) + 1.2(0.84) + 1.2(0.84) = 2.196$, while the best aligned value is 2.172 (the matching $\{13, 24\}$ with s_1^2, s_2^2 in either order; $\{12, 34\}$ gives 2.0 and $\{14, 23\}$ gives 0). The linear program over (13) gives 2.324: the point $x_{12} = 0.19$, $x_{13} = x_{24} = 0.81$, all other $x_e = 0$, is feasible with objective 2.324, and the combination of 0.6 times the inequality for $C = \{1, 2\}$, $B = [4]$ with 0.2 times each of the inequalities for $C = \{1, 2, 3\}$ and $C = \{1, 2, 4\}$, $B = [4]$, bounds the objective by $0.6 \cdot 2 + 0.2 \cdot 2.81 + 0.2 \cdot 2.81 = 2.324$. The value 2.196 is the exact maximum over $O(4)$: with $X = \frac{1}{2}(M_{12} + M_{34}, M_{13} - M_{24}, M_{14} + M_{23})$ and $Y = \frac{1}{2}(M_{12} - M_{34}, M_{13} + M_{24}, M_{14} - M_{23})$ (in the notation of Gil's Appendix A, $X = \frac{1}{2}A_+\hat{v}_+$ and $Y = \frac{1}{2}A_-\hat{v}_-$ with $A_{\pm} = s_1 \pm s_2$ on one orientation class [1, eqs. (A3)–(A6)]) the constraints $\frac{1}{4}\|M\|_F^2 = \|X\|^2 + \|Y\|^2 = 0.905$ and $|\text{Pf } M| = |||X\|^2 - \|Y\|^2| = 0.9$ (an orientation-reversing orthogonal congruence changes the sign of the Pfaffian and exchanges the roles of X and Y , so only the absolute value is invariant on the orbit) give $\{\|X\|^2, \|Y\|^2\} = \{0.9025, 0.0025\}$, the objective is $2(X_1 + Y_1)^2 + 2.4(X_2^2 + Y_2^2)$, symmetric in X and Y , and when $\|Y\|^2 = 0.0025$ it is at most $2.172 - 0.4(X_1 - 5Y_1)^2 + 9.6Y_1^2 \leq 2.172 + 9.6 \cdot 0.0025 = 2.196$, with equality at the displayed matrix. This note does not determine the maximum for general weights.

5 Consequences for the intrinsic description

Corollary 5.1. *Let the populations be fixed.*

1. *With free Youla values, every functional of the form $\sum_{i < j} b_{ij} M_{ij}^2$ with $b_{ij} \geq 0$ is maximized by an aligned state whose support is a maximum-weight matching for the weights b on the support $\{i : a_i > 0\}$ (every entry on an inactive axis is zero, so for positive populations the matching is on all of $[n]$); in particular Gil's hierarchical descriptors that are linear in the squared metaspin entries are extremized in the aligned class.*
2. *For positive populations, the set of achievable coherence weight vectors $(N_{ij}^2/(a_i a_j))_{i < j}$ over all states with these populations has convex hull equal to Edmonds' matching polytope $\mathcal{P}_n$; in*

particular no state can put total squared metaspin weight more than $(|B| - 1)/2$ on the pairs inside an odd set B , whatever its orientation.

3. With Youla values also fixed ($s_1 \leq 1$, read on the support of the populations), the squared cohesion is at most $2 \sum_k s_k^2 a_{2k-1} a_{2k}$, and the bound is attained only if $\|M_{[p],[q]}\|_F^2 = R(p, q)$ for every $p \leq q$ with $a_p > a_{p+1}$ and $a_q > a_{q+1}$ (with $a_{n+1} = 0$), that is, only if every prefix inequality used in the proof is tight on the level sets of the populations. Whether this forces alignment is not examined here.

Proof. (1) and (2) are Theorem 3.1 and Lemma 2.1; (3) is Theorem 4.1 together with the observation that equality in the integrated inequality forces equality in $\Psi \leq h$ for almost every (u, v) , and the prefix pairs (C, B) that occur for a set of (u, v) of positive measure are exactly those named. $\square$

The results here do not determine the image of Gil's Gram map $G(a, Q_M)$ (his question (ii)), or the maximum of a general nonnegative-weight functional $\sum_{i < j} b_{ij} M_{ij}^2$ with prescribed Youla values; Remark 4.5 shows that alignment and the linear program over (13) do not solve the latter in general.

6 Numerical checks

The program `verify_gil.py` (seed 20260929; results in `verification-gil.json`; both published with this note at <https://hypnosmath.org/research/global-maximal-coherence-two-questions-of-gil>) performs the following finite checks. They are implementation controls, not part of the proofs; the searches are local, and the program records the worst shortfall of each search as well as its largest excess, so that a search that fails to reach a bound is visible.

- Free Youla values, $2 \leq n \leq 7$, five population vectors per n (30 instances): 300 random antisymmetric contractions each, and projected gradient ascent over the contraction ball (20 random starts, 4000 steps, the singular values clipped at 1 after each step, a final snap to a partial isometry); the squared cohesion never exceeded (7) beyond roundoff (largest excess $5.6 \cdot 10^{-16}$), the ascent came within 10^{-6} of it in 19 instances and within 10^{-4} in all 30, and its worst shortfall was $7.2 \cdot 10^{-5}$ (at $n = 7$).
- Prescribed Youla values, same dimensions, five spectra per n including $s \equiv 1$: 300 random orientations and 20 Nelder–Mead maximizations, both over $\text{SO}(n)$ through the exponential parametrization (a coordinate sign flip preserves squared entries, so this covers $\text{O}(n)$); the value $2 \sum_k s_k^2 a_{2k-1} a_{2k}$ was not exceeded beyond numerical error (largest excess $1.5 \cdot 10^{-13}$) and was reached within 10^{-6} in all 30 instances.
- Linear programs over the full family (13), over the prefix subfamily alone, over $C = B$ alone, and over $C = B$ with the degree constraints, compared with Theorem 4.1: the first two agree with the theorem to $5.6 \cdot 10^{-17}$ in every instance; the last two exceed it as reported in Remark 4.3.
- Floating-point checks of the odd-set inequalities for random antisymmetric contractions on every odd subset, $3 \leq n \leq 8$ (8760 checks, largest excess -0.032).

7 Provenance

The two questions were found while reviewing the companion manuscript [13], whose hull lemma this note reuses, and the theorems were proved in that session; no step of the mathematics came from the harness. The companion manuscript arose from Hypnos [14], a research harness built at the direction of David Ross. Its proposal models are Gemma 4 31B, a dense model in 4-bit activation-aware weight quantization (AWQ), `QuantTrio/gemma-4-31B-it-AWQ`; Gemma 4 26B, a mixture-of-experts model with 4 billion active parameters, served in AWQ from a quantization of `google/gemma-4-26B-A4B-it`; and Qwen3 32B, a dense model in 4-bit AWQ, `Qwen/Qwen3-32B-AWQ`. They run through vLLM on hardware operated by Ross and generate claims from a notebook of structured records. Claude Opus (Anthropic), accessed through a subscription, reviews samples under pre-registered rules; claims selected for execution include a computation that the harness runs. Proposals, verdicts and execution records are retained in the private repository and database. The site <https://hypnosmath.org> publishes reviewed explanations and selected computational records. The program of Section 6 and its recorded outputs are published with this note at <https://hypnosmath.org/research/global-maximal-coherence-two-questions-of-gil>, with the note's source and its reviews.

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