

Approximate Antiunitary Symmetry as a Matching Problem: Audit, Sharpenings, and Extensions

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Abstract

This note accompanies an adversarial review of the manuscript *Approximate Antiunitary Symmetry as a Matching Problem* (commit 9820d1ed). The manuscript's main theorem, its lemma on squared entries of skew-symmetric contractions, and all stated corollaries are correct. We record the sharpenings found (rigidity needs only uniqueness; the hard-constraint corollary needs no sign condition) and prove several extensions: an exact identity with diagonal (vertex) costs, which yields the matching formula under the condition $2c_{ij} \geq c_{ii} + c_{jj}$ and fails outside it with an explicit continuous minimizer; a quantitative Kramers inequality in every unitarily invariant norm; a classical derivation of the single-observable case from Wigner's normal form, Kramers' theorem and Mirsky's inequality, together with a tuple example showing why the polytope argument is needed; the failure of the Shor semidefinite relaxation to see the parity obstruction; and two theorems answering open questions posed by J. J. Gil (Entropy 28:877, 2026) on the maximal coherence of density matrices with prescribed intrinsic populations, with and without prescribed Youla values. Original claims of the manuscript are cited as such; everything proved here is the reviewer's.

1 Setting and what the manuscript proves

For $U \in \mathbf{U}(n)$ put $S = \frac{1}{2}(U + U^T)$ and $K = \frac{1}{2}(U - U^T)$ (plain transpose), so $S^T = S$, $K^T = -K$, $\|S\|_F^2 + \|K\|_F^2 = n$, and both are contractions. Since $U^T \bar{U} = (U^* U)^T = I$,

$$U \bar{U} + I = 2S \bar{U}, \quad \|U \bar{U} + I\|_F^2 = 4\|S\|_F^2. \quad (1)$$

Write $x_{ij} = |K_{ij}|^2$ for $i < j$ and let $\mathcal{P}_n$ be Edmonds' matching polytope of the complete graph [1]. The manuscript proves:

Lemma 1.1 (Manuscript, Lemma 2.2). *If $K^T = -K$ and $\|K\|_{\text{op}} \leq 1$ then $x \in \mathcal{P}_n$, and $\text{conv}\{x(K)\} = \mathcal{P}_n$.*

Theorem 1.2 (Manuscript, Theorem 2.1). *For real symmetric $c \geq 0$ with $c_{ii} = 0$ and $\tau \geq 0$,*

$$\min_{U \in \mathbf{U}(n)} \left[\sum_{i,j} c_{ij} |U_{ij}|^2 + \tau \|U \bar{U} + I\|_F^2 \right] = \min_{M \in \mathcal{M}_n} \left[2 \sum_{\{i,j\} \in M} c_{ij} + 4\tau(n - 2|M|) \right],$$

attained by the real signed-swap matrix of an optimal matching M .

The proof rests on the exact identity

$$E_{c,\tau}(U) = 4\tau n + 2 \sum_{i < j} (c_{ij} - 4\tau)x_{ij} + 2 \sum_{i < j} c_{ij}|S_{ij}|^2, \quad (2)$$

Lemma 1.1, and the fact that an affine function on $\mathcal{P}_n$ is minimized at a matching vertex. We verified every step independently (review file `01-proof-reconstruction.md`); the odd-set inequalities follow because an odd-order principal skew block is singular and its squared Frobenius norm is at most its rank. Nonnegativity of c is used exactly once, to drop the last sum in (2).

2 Vertex penalties: the diagonal-cost identity and its sharp limit

Theorem 2.1. *Let c be real symmetric (any sign, diagonal allowed), $d_i = c_{ii}$, $w_{ij} = 2c_{ij} - d_i - d_j$ for $i \neq j$, and $\tau \geq 0$. For every $U \in \mathbf{U}(n)$,*

$$E_{c,\tau}(U) = \sum_i d_i + 4\tau n + \sum_{i < j} (w_{ij} - 8\tau)x_{ij} + \sum_{i < j} w_{ij}|S_{ij}|^2. \quad (3)$$

If $w_{ij} \geq 0$ for all $i \neq j$, then

$$\min_{U \in \mathbf{U}(n)} E_{c,\tau}(U) = \min_{M \in \mathcal{M}_n} \left[2 \sum_{\{i,j\} \in M} c_{ij} + \sum_{i \notin V(M)} (c_{ii} + 4\tau) \right], \quad (4)$$

attained by the signed-swap matrix of an optimal M .

Proof. Splitting the cost sum and using $|U_{ij}|^2 + |U_{ji}|^2 = 2|S_{ij}|^2 + 2x_{ij}$, $\sum_{i,j} c_{ij}|U_{ij}|^2 = \sum_i d_i|U_{ii}|^2 + 2 \sum_{i < j} c_{ij}(|S_{ij}|^2 + x_{ij})$. Adding the i -th row and column norms of U gives $|U_{ii}|^2 = 1 - \sum_{j \neq i} (|S_{ij}|^2 + x_{ij})$, hence $\sum_i d_i|U_{ii}|^2 = \sum_i d_i - \sum_{i < j} (d_i + d_j)(|S_{ij}|^2 + x_{ij})$. With (1), $\tau\|U\bar{U} + I\|_F^2 = 4\tau n - 8\tau \sum_{i < j} x_{ij}$. Collecting terms gives (3). Under $w \geq 0$ the last sum is nonnegative, $x \in \mathcal{P}_n$ by Lemma 1.1, and the affine part is minimized at a vertex $\mathbf{1}_M$, where it equals the bracket in (4); the signed-swap matrix has $x = \mathbf{1}_M$ and diagonal S , so it attains that value. $\square$

Proposition 2.2 (Sharpness). *For $n = 3$, $c = I$ (so $w_{ij} = -2$) and every $\tau \geq 0$, the rotation U_θ about $u = (1, 1, 1)/\sqrt{3}$ with $\cos \theta = -1/(2 + 12\tau)$ satisfies*

$$E_{c,\tau}(U_\theta) = 4\tau + \frac{2\tau}{1 + 6\tau} < 1 + 4\tau = \text{right side of (4)}.$$

Proof. $U_\theta = \cos \theta I + (1 - \cos \theta)uu^T + \sin \theta [u]_\times$ is real orthogonal with all diagonal entries $(1 + 2\cos \theta)/3$ and symmetric part $S = \cos \theta I + (1 - \cos \theta)uu^T$, whose eigenvalues are $\cos \theta, \cos \theta, 1$. Writing $\gamma = \cos \theta$, $E = (1 + 2\gamma)^2/3 + 4\tau(2\gamma^2 + 1)$, minimized at $\gamma = -1/(2 + 12\tau)$, where $1 + 2\gamma = 6\tau/(1 + 6\tau)$ and the value is $12\tau^2/(1 + 6\tau)^2 + 4\tau + 2\tau/(1 + 6\tau)^2 = 4\tau + 2\tau/(1 + 6\tau)$. Every matching leaves a vertex uncovered at cost $1 + 4\tau$, and $2\tau/(1 + 6\tau) < 1/3$. $\square$

Numerically the rotation family is the global minimizer for this instance: multistart minimization over $\mathbf{U}(3)$ agrees with $4\tau + 2\tau/(1 + 6\tau)$ to 10^{-12} at ten values of τ (review file `exp04`). The mechanism is that the symmetric part S obeys the degree constraints but no odd-set constraint, so once symmetric pairing is cheaper than leaving vertices uncovered, a fractional odd cycle wins and the answer is no longer combinatorial. Physically, a commuting T -odd observable Q (which should anticommute with the antiunitary) contributes $(q_i + q_j)^2|W_{ij}|^2$ and hence a diagonal cost $4q_i^2$; the condition $w \geq 0$ reads $\sum_r (h_i^r - h_j^r)^2 \geq \sum_s (q_i^s - q_j^s)^2$ for every pair.

3 Quantitative Kramers pairing in every unitarily invariant norm

Let $n = 2m$, H Hermitian with eigenvalues $\lambda_1 \leq \dots \leq \lambda_n$, $\delta_j = \lambda_{2j} - \lambda_{2j-1}$. For an antiunitary $T = UC$, $THT^{-1} = U\overline{H}U^*$, and $T^2 = -1$ iff $U^T = -U$.

Theorem 3.1. *For every antiunitary T with $T^2 = -1$ and every unitarily invariant norm $\|\cdot\|$,*

$$\|H - THT^{-1}\| \geq \|\text{diag}(\delta_1, \delta_1, \dots, \delta_m, \delta_m)\|,$$

with equality for the signed swap on consecutive eigenvectors. In particular $\|H - THT^{-1}\|_p \geq (2\sum_j \delta_j^p)^{1/p}$ and $\|H - THT^{-1}\|_{\text{op}} \geq \max_j \delta_j$. Moreover, for every unitarily invariant norm,

$$\min\{\|H - H'\| : H' \text{ Hermitian with all multiplicities even}\} = \|\text{diag}(\frac{\delta_1}{2}, \frac{\delta_1}{2}, \dots, \frac{\delta_m}{2}, \frac{\delta_m}{2})\|.$$

Proof. Let $\tilde{H} = \frac{1}{2}(H + THT^{-1})$. Because $T^2 = -1$ is central, $T\tilde{H}T^{-1} = \tilde{H}$. If $\tilde{H}\psi = \mu\psi$ then $\tilde{H}(T\psi) = \mu T\psi$ and $\langle \psi, T\psi \rangle = 0$ (antiunitarity gives $\langle T\psi, T^2\psi \rangle = \langle \psi, T\psi \rangle$, and $T^2 = -1$ turns this into $\langle T\psi, \psi \rangle = -\langle T\psi, \psi \rangle$), so every eigenspace of $\tilde{H}$ carries an antiunitary with square -1 and is even-dimensional (Kramers). Let $\rho_1 \leq \dots \leq \rho_n$ be its eigenvalues; then $\rho_{2j-1} = \rho_{2j}$.

Mirsky's theorem [2] gives, for any unitarily invariant norm, $\|\text{diag}(\lambda - \rho)\| \leq \|H - \tilde{H}\| = \frac{1}{2}\|H - THT^{-1}\|$. Let $v_i = |\lambda_i - \rho_i|$ and $u = (\delta_1/2, \delta_1/2, \dots, \delta_m/2, \delta_m/2)$. For each pair, $v_{2j-1} + v_{2j} \geq \delta_j$ by the triangle inequality, so $\max(v_{2j-1}, v_{2j}) \geq \delta_j/2$. Fix $k = 2l + r$, $r \in \{0, 1\}$, and let $\delta_{(1)} \geq \delta_{(2)} \geq \dots$ be the sorted gaps. The sum of the k largest entries of u is $\sum_{j \leq l} \delta_{(j)} + r\delta_{(l+1)}/2$; the sum of the k largest entries of v is at least the sum of v over the $2l$ indices of the top- l pairs plus, if $r = 1$, the larger entry of the next pair, which is at least the same. Thus $u \prec_w v$, and by Ky Fan's dominance theorem [3, Thm. IV.2.2 and Cor. IV.2.3] $\|\text{diag}(u)\| \leq \|\text{diag}(v)\| = \|\text{diag}(\lambda - \rho)\|$. Hence $\|H - THT^{-1}\| \geq 2\|\text{diag}(u)\| = \|\text{diag}(\delta_1, \delta_1, \dots)\|$.

Equality: in an eigenbasis, the signed swap U_0 on $(2j-1, 2j)$ has $U_0^T = -U_0$ and $U_0\overline{H}U_0^* = U_0HU_0^T$ swaps each pair of diagonal entries, so $H - U_0HU_0^{-1} = \text{diag}(-\delta_1, \delta_1, \dots)$.

For the distance statement, the lower bound is the same argument with any even-multiplicity H' in place of $\tilde{H}$, and the midpoint replacement in an eigenbasis attains it. Matrices with even multiplicities are exactly those commuting with some antiunitary of square -1 (Kramers; conversely the signed swap in an eigenbasis). $\square$

The Frobenius case is the manuscript's Corollary 3.5 with Proposition 4.2 for one observable. For a single cluster in Frobenius norm the distance statement is in Pinter, Frank, Varjas and Pályi [8, Thm. 3.2.2]; the all-pairs, all-norms form with the pairing optimized appears to be new. The bound uses the spectrum only, so for a *prescribed* physical T it is a certified lower bound on symmetry breaking; in a synthetic spin-orbit ring with a Zeeman field the operator-norm bound is within 0.05% of the physical commutator at weak field (review file `exp08`).

4 The single-observable case is classical; the tuple case is not

Proposition 4.1. (a) *For one Hermitian matrix, Theorem 1.2 with $c_{ij} = (\lambda_i - \lambda_j)^2$ follows from Wigner's normal form [5, 6], Kramers' theorem and the Hoffman-Wielandt inequality, for every n and $\tau \geq 0$. (b) For commuting tuples this route fails: for $H_1 = \text{diag}(0, 0, 1, 1)$, $H_2 = \text{diag}(0, 1, 0, 1)$ and $T^2 = -1$, every single-observable bound is 0 while the joint minimum is 4.*

Proof. (a) Let $V = U\overline{U} = T^2$. Its spectrum consists of $+1$ (multiplicity m_+), -1 (multiplicity $2m_-$, even, since T restricted to that eigenspace has square -1), and pairs $e^{\pm i\theta}$, $0 < \theta < \pi$, whose

eigenspaces have equal dimension m_θ because T maps one onto the other. The symmetrized $\tilde{H}$ commutes with T and T^2 and preserves these eigenspaces; on the -1 eigenspace its eigenvalues have even multiplicity by Kramers, and on $E_\theta \oplus E_{-\theta}$ they do too because T intertwines the two restrictions antiunitarily. With $P = m_- + \sum_\theta m_\theta$ we have $n = m_+ + 2P$, and the sorted spectrum ρ of $\tilde{H}$ contains P disjoint consecutive equal pairs. Hoffman–Wielandt gives $\|H - \tilde{H}\|_F^2 \geq \sum_i (\lambda_i - \rho_i)^2 \geq \frac{1}{2} \min_{|M|=P} \sum_{\{i,j\} \in M} (\lambda_i - \lambda_j)^2$, and $\tau \|T^2 + I\|_F^2 = \tau \sum_\mu |\mu + 1|^2 \geq 4\tau m_+ = 4\tau(n - 2P)$. Summing, $E \geq \min_M [2 \sum_M (\lambda_i - \lambda_j)^2 + 4\tau(n - 2|M|)]$, which the signed swap attains.

(b) Each H_r has doubly degenerate spectrum, so Theorem 3.1 gives 0 for each. The manuscript’s Corollary 3.5 gives $2 \min_{\text{perfect } M} \sum \|\lambda_i - \lambda_j\|^2 = 4$ for the joint vectors $(0,0), (0,1), (1,0), (1,1)$. The symmetrized $\tilde{H}_1, \tilde{H}_2$ need not commute, so there is no joint spectrum to which a Mirsky-type inequality applies. $\square$

5 The Shor relaxation misses parity

Lift $u = \text{vec}(U)$ to $Z = uu^*$; the objective $\sum c_{ij} |U_{ij}|^2 + \tau \|U + U^T\|_F^2$ (equal to $E_{c,\tau}$ on $U(n)$) and the constraints $U^*U = I = UU^*$ are linear in Z . The Shor relaxation keeps $Z \succeq 0$ and drops the rank condition.

Proposition 5.1. *For odd n , $c = 0$ and $\tau > 0$, the Shor relaxation has value 0 while the true minimum is 4τ . A certificate is $Z_{(ij),(ij)} = \frac{1}{n-1}$, $Z_{(ij),(ji)} = -\frac{1}{n-1}$ for $i \neq j$, all other entries 0.*

Proof. Z is block diagonal with 2×2 blocks $\frac{1}{n-1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, so $Z \succeq 0$. The constraint $\sum_i \overline{U_{ik}} U_{il} = \delta_{kl}$ reads $\sum_i Z_{(il),(ik)} = \delta_{kl}$: for $k = l$ it is $\sum_{i \neq k} \frac{1}{n-1} = 1$, and for $k \neq l$ every term vanishes because $(i, l), (i, k)$ are never a transposed pair; likewise for $UU^* = I$. The objective is $\tau \sum_{i \neq j} (\frac{1}{n-1} + \frac{1}{n-1} - \frac{2}{n-1}) = 0$. $\square$

In the manuscript’s language this is the fractional matching $x_{ij} = 1/(n-1)$ on every pair, which satisfies all degree constraints and violates the odd-set constraint on $[n]$. A convex relaxation that keeps only quadratic consequences of unitarity cannot see the rank argument behind the odd-set inequalities; the exact answer needs Edmonds’ theorem or an equivalent parity argument.

6 Rigidity needs only uniqueness

Proposition 6.1. *In the manuscript’s Proposition 3.2 the hypothesis $c_{ij} > 0$ for all $i \neq j$ is redundant: uniqueness of the optimal matching M_* already forces every minimizer to be a phased signed swap.*

Proof. At a minimizer $x = \mathbf{1}_{M_*}$ (uniqueness) and $\sum_{i < j} c_{ij} |S_{ij}|^2 = 0$. Rows and columns of matched vertices are forced by $|K_{ij}| = 1$. If two uncovered vertices i, j had $c_{ij} = 0$, the matching $M_* \cup \{i, j\}$ would cost $\text{cost}(M_*) - 8\tau \leq \text{cost}(M_*)$, contradicting uniqueness ($\tau = 0$) or optimality ($\tau > 0$). So $c_{ij} > 0$ on uncovered pairs, $S_{ij} = 0$ there, and the uncovered block is diagonal with unimodular entries. $\square$

Uniqueness cannot be dropped: for $n = 3$, $c = 0$, $\tau = 1$ the real orthogonal matrix with rows $(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}), (-\frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{1}{2})$ has $x = (\frac{1}{2}, \frac{1}{2}, 0)$, $\|S\|_F^2 = 1$, energy 4, and is not a signed swap.

7 Two open questions of Gil

Gil [11] writes a density matrix in its intrinsic reference basis as $\rho = A + iN$ with $A = \text{diag}(a_1 \geq \dots \geq a_n \geq 0)$ and N real antisymmetric, defines the metaspin tensor $M_{ij} = N_{ij}/\sqrt{a_i a_j}$ (his eq. (7)), notes $\rho = A^{1/2}(I + iM)A^{1/2}$ (his eq. (8)) so that positivity is equivalent to all Youla values s_k of M being at most 1, i.e. $\|M\|_{\text{op}} \leq 1$, and calls $\|N\|_{\text{F}}^2 = 2 \sum_{i < j} a_i a_j M_{ij}^2$ (his eq. (33)) the cohesion. His Theorem 2 shows that over the “aligned class” (M supported on a matching of axes) the maximal cohesion for fixed populations is $2 \sum_k a_{2k-1} a_{2k}$; Section 8.2 asks (i) for the global maximum with fixed populations and fixed Youla values over all orientations, and (iii) whether the adjacent-pairing maximum is globally optimal beyond the aligned class.

Theorem 7.1 (answer to (iii)). *For fixed populations, the maximum of $\|N\|_{\text{F}}^2$ over all density matrices with those intrinsic populations is $2 \sum_{k=1}^{\lfloor n/2 \rfloor} a_{2k-1} a_{2k}$, attained in the aligned class. More generally, for nonnegative weights b_{ij} , $\max \sum_{i < j} b_{ij} M_{ij}^2$ over admissible metaspin tensors is the maximum weight of a matching.*

Proof. For $a > 0$ the admissible M are exactly the real skew contractions, so $x = (M_{ij}^2) \in \mathcal{P}_n$ by Lemma 1.1; a nonnegative linear functional on $\mathcal{P}_n$ is maximized at a matching vertex, and for the weights $a_i a_j$ the best matching is the adjacent one by the exchange inequalities $a_i a_j + a_k a_l - (a_i a_k + a_j a_l) = (a_i - a_l)(a_j - a_k) \geq 0$ and $a_i a_j + a_k a_l - (a_i a_l + a_j a_k) = (a_i - a_k)(a_j - a_l) \geq 0$ for $a_i \geq a_j \geq a_k \geq a_l$ (Gil’s (26), (27)). The adjacent signed swap is a skew contraction, so the value is attained. Zero populations force the corresponding rows and columns of N to vanish and reduce to the support. $\square$

Theorem 7.2 (answer to (i)). *Let $a_1 \geq \dots \geq a_n \geq 0$, $m = \lfloor n/2 \rfloor$, $s_1 \geq \dots \geq s_m \geq 0$, and let M be any real or complex skew-symmetric matrix with singular values $s_1, s_1, \dots, s_m, s_m$ (and 0 if n is odd). Then*

$$\sum_{i < j} a_i a_j |M_{ij}|^2 \leq \sum_{k=1}^m s_k^2 a_{2k-1} a_{2k},$$

with equality for the aligned adjacent configuration $M_{2k-1, 2k} = s_k$. Hence for fixed populations and fixed Youla values the global maximum of the cohesion over all orientations is $2 \sum_k s_k^2 a_{2k-1} a_{2k}$.

Proof. Write $x_{ij} = |M_{ij}|^2$. For nonnegative numbers $a_i a_j = \int_0^\infty \int_0^\infty \mathbf{1}[a_i > u] \mathbf{1}[a_j > v] du dv$, and symmetrizing in (u, v) ,

$$\sum_{i < j} a_i a_j x_{ij} = \iint \Psi(u, v) du dv, \quad \Psi(u, v) = \frac{1}{2} \sum_{i < j} x_{ij} (\mathbf{1}[a_i > u, a_j > v] + \mathbf{1}[a_i > v, a_j > u]),$$

while the right side equals $\iint h$ with $h(u, v) = \frac{1}{2} \sum_k s_k^2 (\mathbf{1}[a_{2k-1} > u, a_{2k} > v] + \mathbf{1}[a_{2k-1} > v, a_{2k} > u])$. Both integrands are symmetric, so it suffices to show $\Psi \leq h$ for $u \geq v$. Put $C = \{i : a_i > u\} = \{1, \dots, p\}$ and $B = \{i : a_i > v\} = \{1, \dots, q\}$, prefix sets with $C \subseteq B$. For $i < j$, $a_i \geq a_j$, so $\mathbf{1}[a_i > u, a_j > v] = \mathbf{1}[i \in C, j \in B]$ and $\mathbf{1}[a_i > v, a_j > u] = \mathbf{1}[j \in C]$. Hence $2\Psi = x(E(C)) + x(C, B \setminus C) + x(E(C))$, where $x(C, B \setminus C)$ is the mass on pairs between C and $B \setminus C$; in matrix terms $2x(E(C)) = \|M_C\|_{\text{F}}^2$ and $x(C, B \setminus C) = \|M_{C, B \setminus C}\|_{\text{F}}^2$, so

$$\Psi(u, v) = \frac{1}{2} \|M_{C, B}\|_{\text{F}}^2,$$

the squared Frobenius norm of the block with rows C and columns B . Similarly $\mathbf{1}[a_{2k-1} > u, a_{2k} > v] = \mathbf{1}[2k-1 \leq p, 2k \leq q]$ and $\mathbf{1}[a_{2k-1} > v, a_{2k} > u] = \mathbf{1}[2k \leq p]$, so, s being sorted,

$$h(u, v) = \frac{1}{2} R(p, q), \quad R(p, q) = \sum_{k \leq \min(\lceil p/2 \rceil, \lfloor q/2 \rfloor)} s_k^2 + \sum_{k \leq \lfloor p/2 \rfloor} s_k^2.$$

It remains to prove $\|M_{C,B}\|_F^2 \leq R(p, q)$. The block consists of p rows of the principal submatrix M_B , which is skew-symmetric of order q with singular values $t_1, t_1, \dots, t_r, t_r$, $r = \lfloor q/2 \rfloor$ (and a zero if q is odd). Ky Fan's maximum principle gives $\|M_{C,B}\|_F^2 = \text{tr}(PM_B M_B^* P) \leq \sum_{k \leq p} \sigma_k(M_B)^2$ for the rank- p coordinate projection P , and interlacing for singular values of submatrices [4, Thm. 3.1.2, Cor. 3.1.3] gives $t_k \leq s_k$. If p is even, $\sum_{k \leq p} \sigma_k(M_B)^2 = 2 \sum_{k \leq p/2} t_k^2 \leq 2 \sum_{k \leq p/2} s_k^2 = R(p, q)$ since $p/2 \leq \lfloor q/2 \rfloor$. If p is odd and $p < q$, then $(p+1)/2 \leq \lfloor q/2 \rfloor$ and $\sum_{k \leq p} \sigma_k(M_B)^2 = 2 \sum_{k \leq (p-1)/2} t_k^2 + t_{(p+1)/2}^2 \leq R(p, q)$. If $p = q$ is odd, $\sum_{k \leq p} \sigma_k(M_B)^2 = 2 \sum_{k \leq r} t_k^2 \leq 2 \sum_{k \leq (q-1)/2} s_k^2 = R(p, q)$. Integrating $\Psi \leq h$ proves the inequality; equality at the aligned configuration is immediate. $\square$

Remark 7.3. The proof never uses that C, B are prefix sets, so it yields the valid inequalities $2x(E(C)) + x(C, B \setminus C) \leq R(|C|, |B|)$ for all $C \subseteq B$; $|C| = 1$ gives the degree constraints and $C = B$ the refined odd/even-set constraints. The coarser family with $C = B$ only is not sufficient for product weights (its LP maximum exceeds the true value by up to 0.045 in random instances), whereas the LP over the full family equals the value of Theorem 7.2 to 10^{-17} (review file exp03b). The skew-symmetric Schur–Horn results of Sanyal, Sottile and Sturmfels [9, Prop. 3.10] and of Tam [10] describe one linear entry per 2×2 block under congruence; Theorem 7.2 concerns the squared moduli of all off-diagonal entries and was not found there. The generic Hermitian bound $\|A^{1/2} X A^{1/2}\|_F^2 \leq \sum_i a_i^2 \mu_i^2$ applied to $X = iM$ gives only $\sum_k s_k^2 (a_{2k-1}^2 + a_{2k}^2)$, weaker by the AM–GM inequality; the zero diagonal of a skew matrix is what allows the product form.

8 Normal matrices and the two symmetry relations

Proposition 8.1. *Let V be normal with eigenvalues ν_i . (a) For the inverse relation $TVT^{-1} = V^*$ (for unitary V , the Floquet time-reversal relation $TVT^{-1} = V^{-1}$), $\min_U \|V^* U - U \bar{V}\|_F^2 + \tau \|U \bar{U} + I\|_F^2 = \min_M [2 \sum_{\{i,j\} \in M} |\nu_i - \nu_j|^2 + 4\tau(n - 2|M|)]$. (b) For the commuting relation $TVT^{-1} = V$ the costs are $c_{ij} = |\nu_i - \bar{\nu}_j|^2$ with diagonal $c_{ii} = 4(\text{Im } \nu_i)^2$, and Theorem 2.1 applies iff $|\text{Re } \nu_i - \text{Re } \nu_j| \geq |\text{Im } \nu_i - \text{Im } \nu_j|$ for all pairs.*

Proof. With $V = QDQ^*$ and $U = QWQ^T$, $V^* U - U \bar{V} = Q(\bar{D}W - W\bar{D})Q^T$ has entries $(\bar{\nu}_i - \bar{\nu}_j)W_{ij}$, and $VU - U\bar{V} = Q(DW - W\bar{D})Q^T$ has entries $(\nu_i - \bar{\nu}_j)W_{ij}$; apply Theorem 1.2, respectively Theorem 2.1. $\square$

9 What remains open

(1) A combinatorial description of the minimum in Theorem 2.1 when $w \not\geq 0$; Proposition 2.2 shows continuous minimizers occur. (2) The convex hull of the joint vectors $(|S_{ij}|^2, |K_{ij}|^2)$ over $U(n)$; the projection to x is $\mathcal{P}_n$, but the joint set is strictly smaller than the obvious relaxation: by Katz's description of the extreme points of the symmetric doubly stochastic polytope and the observation of O'Meara and Pereira [7] that odd-cycle extreme points are notunistochastic, the symmetric part alone already has a hull strictly inside its degree polytope for $n \geq 3$. (3) For noncommuting tuples the manuscript's sandwich bound needs a commuting approximant; joint approximate diagonalization gives one, and the projection bound $\Phi_\tau(\mathbf{H}) \geq \sup_{|t|=1} \Phi_\tau(t \cdot \mathbf{H})$ (Cauchy–Schwarz, each term solved by the sorted recurrence) is a complementary certified lower bound; neither is exact (review file exp07). (4) Whether Theorem 7.2 extends from product weights $a_i a_j$ to general nonnegative weights with prescribed singular values.

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